Vectors

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Fixed Lines, Eigenvectors and Eigenvalues

Vectors

The origin is fixed under any linear transformation. For most cases there are lines that are also fixed.

Definition 4

A line l is fixed under a linear transformation t of $\mathbb{R}^n$ if t maps any point X on l to a point X' also on l.

X' and X are not nesessarily the same point, although this is a possibility.

Definition 5

Let t be a linear transformation of $\mathbb{R}^n$ with it's corresponding matrix A.

Then if $Av = \lambda v$

v is called an eigenvector of t with corresponding eigenvalue λ of t. Sometimes these are referred to as the eigenvector and corresponding eigenvalue of matrix A.

Theorem 2

If v is an eigenvector of a linear transformation t of $\mathbb{R}^n$, with corresponding eigenvalue λ.

For any non-zero real number k, then kv is an eigenvector of t with corresponding eigenvalue λ.

Proof

Let A be the matrix of t.
Then $Av = \lambda v$

so that $t(kv) = A(kv) = k(Av) = k(\lambda v) = \lambda(kv)$

Note! 'Inside an Eigenvector!'

If v is an eigenvector, with corresponding eigenvalue of λ, of linear transformation t of $\mathbb{R}^n$.
Then the line l which passes through the origin and the point V, with position vector v, is a fixed line of t.
This line of t and the position vector of any point on l, other than O (origin), is an eigenvector of t, with an eigenvalue of λ.

- Proof

Any point on l must have a position vector kv for some $k \in \mathbb{R}$, and the result follows from Theorem 2.

This means that on a fixed line l, λ is the scale factor by which all distances from the origin are multiplied under the action of t.

Ex. Example

- Suppose t is a linear transformation of $\mathbb{R}^2$ with matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Show that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of t, and find their values.
- Next we will verify that $\begin{pmatrix} k \\ k \end{pmatrix}$ is also an eigenvector of t with the same eigenvalue $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $y=x$ is a fixed line of t.
- Finally we will find an equation for another fixed line of t.

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→ **Solution** to prevous page's example

$$\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\3\end{pmatrix}=3\begin{pmatrix}1\\1\end{pmatrix}$$

So $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector with a value of 3.

So continuing $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}=(-1)\begin{pmatrix}1\\-1\end{pmatrix}$

Here we have $\begin{pmatrix}1\\-1\end{pmatrix}$ so this is an eigenvector with eigenvalue (-1)

Next... $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}k\\k\end{pmatrix}=\begin{pmatrix}3k\\3k\end{pmatrix}=3\begin{pmatrix}k\\k\end{pmatrix}$

So $\begin{pmatrix}k\\k\end{pmatrix}$ is an eigenvector with eigenvalue 3 now for any $k \in \mathbb{R}$.

A point lies on $y=x$ only if its position vector is of form $\begin{pmatrix}k\\k\end{pmatrix}$

finding the point on $y=x$

Taking the form $\begin{pmatrix}k\\k\end{pmatrix}$ to find a point on $y=x$, we know that any such vector we need will be mapped to another position vector of another point on $y=x$.

- So $y=x$ is a fixed line of t, since a similar result can be seen for $\begin{pmatrix}k\\-k\end{pmatrix}$ which is the fixed line of $y=-x$.

Consider the linear transformation t whose matrix is $\begin{pmatrix}a&b\\c&d\end{pmatrix}$. Then if l is a fixed line (of t) passing through the origin, and (x,y) is a point on l for some $\lambda \in r$ we have

$$\lambda\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \quad \text{or} \quad \begin{aligned}\lambda x&=ax+by\\ \lambda y&=cx+dy\end{aligned}$$

By eliminating x and y we get $\underline{\lambda^2-(a+d)\lambda+(ad-bc)=0}$

(continued) From the theory of quadratic equations, we know that if α and β are the roots of the quadratic equation

$$px^2 + qx + r = 0$$

then the product of the roots is equal to r/p, and the sum of the roots is equal to $-q/p$.

So in the final equation of the previous page $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$ we see that the product of the roots is $ad-bc$, which means that the product of the eigenvalues of a linear transformation t of $\mathbb{R}^2$ is equal to the determinant of the matrix of t.

Because, when the eigenvalues are distinct and real, they give the scale factors of t in two given directions.
Their product gives the scale factor of the effect of t on any area.

Even if the eigenvalues are not real, or distinct. The determinant of the matrix of t still gives the scale factor of the effect of t on any area.

REVIEW

Quick review of eigenvectors and eigenvalues

- Suppose t is a linear transformation of $\mathbb{R}^n$ with matrix A, and $Av = \lambda v$ - then v is an eigenvector of t, and λ it's eigenvalue.
- The characteristic equation for a square matrix A is $|A - \lambda I| = 0$ the solutions to which are eigenvalues of A.
- Eigenvectors of a linear transformation determine the fixed lines directions through the origin. The corresponding eigenvalues give the scale factor of the transformation, along the fixed lines

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Ex.

Example

1. Find the eigenvalues and eigenvectors of the transformation of $\mathbb{R}^2$ whose matrix is $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

and so illustrate the transformation with a diagram.

2. Repeat the process but with the transformation matrix $\begin{pmatrix} 3 & 7 \\ 2 & -2 \end{pmatrix}$

Solution

SOLUTION - PART ONE

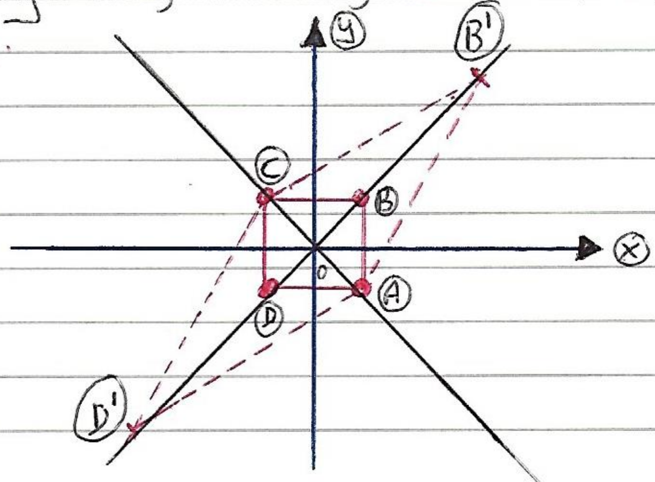
1. By using equation $\lambda^2-(a+d)\lambda+(ad-bc)=0$ from the previous pages, we find the eigenvalues of the transformation are the solutions to the equation.

$$\lambda^2-6\lambda+5=0$$

The solutions are $\lambda=1$ and $\lambda=5$, and by substituting these values into equations $\lambda x = ax + by$ and $\lambda y = cx + dy$ we find that the equation of the fixed line for $\lambda=1$ is $y=-x$

We also find that for $\lambda=5$, $y=x$.

So our transformation keeps distances in the direction of the line $y=-x$ fixed, and stretches distances by a factor of 5 in the direction of $y=x$.



Solution

SOLUTION - PART TWO

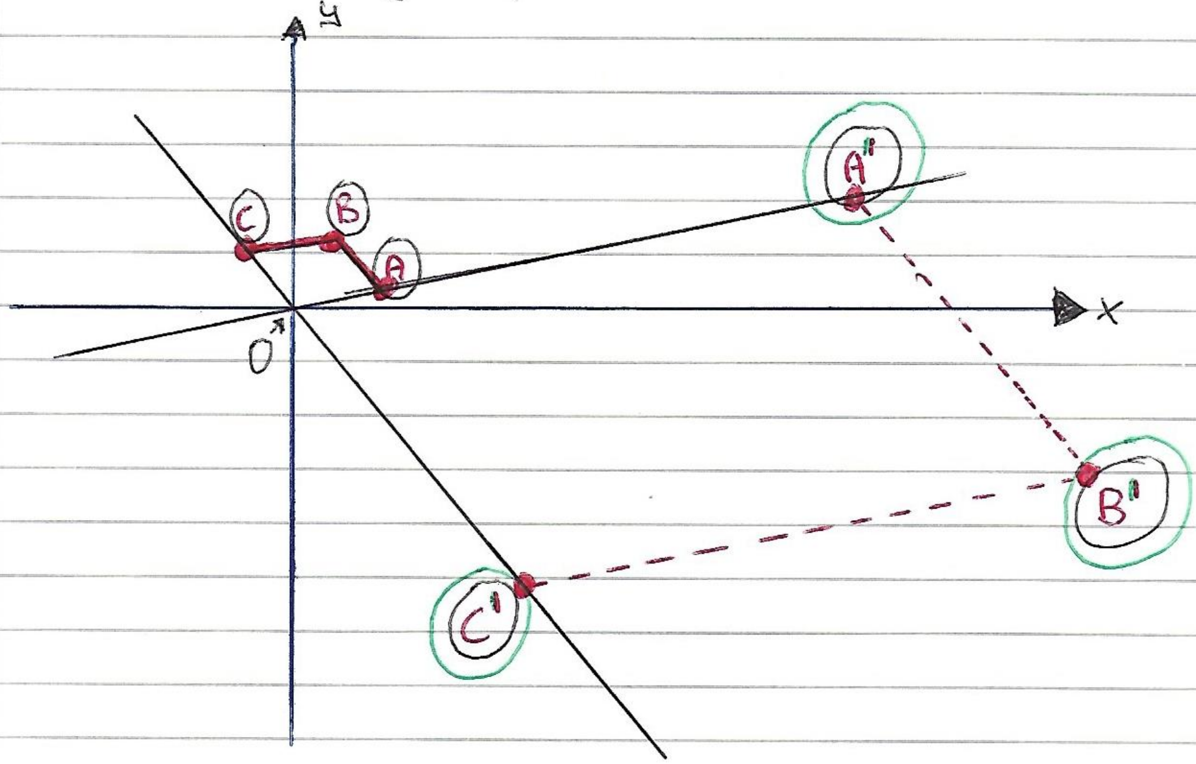
② In the second case from the other side of this page, we use equation

$$\lambda^2-(a+d)\lambda+(ad-bc)=0$$

again, Finding $\lambda=-4$ and $\lambda=5$, and the corresponding fixed lines are respectively $y=-x$ and $7y=2x$.

Thus the distances in the direction of $y=-x$ are multiplied by -4.

This means there is a reflection of the line in the origin as well as a scaling by a factor of 4, while in the direction of $7y=2x$ there is simply a scaling by a factor of 5. See below diagram.



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Fixed lines, Eigenvectors and Eigenvalues	Vectors

Eigenvectors and eigenvalues in special cases

Rotation about O

In this case $a = d = \cos\theta$ and $-b = c = \sin\theta$, so that $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$ becomes

$$\lambda^2 - 2\cos\theta\lambda + \cos^2\theta + \sin^2\theta = 0$$

which can be written as

$$(\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

If $\sin\theta \neq 0$ this has no real solutions, and the rotation has no fixed lines. But if $\sin\theta = 0$, then $\lambda = \cos\theta = \pm 1$ and our rotation is through a multiple of π.

If θ is an even multiple of π then $\lambda = 1$, but if θ is an odd multiple of π then $\lambda = -1$.

In both cases the determinant of the matrix is 1, as we should expect. Since a rotation leaves areas the same.

Reflection in a line through Origin

Reflection in a Line through Origin

Suppose the line makes an angle θ with the x axis (measured anticlockwise).

In this case $a = -d = \cos 2\theta$ and $b = c = \sin 2\theta$. So our equation we have been using becomes

$$\lambda^2 - \cos^2 2\theta - \sin^2 2\theta = 0$$

which reduces to $\lambda^2 - 1 = 0$

So λ can take either the value 1 or -1.
If $\lambda = 1$, from previous equations $\lambda x = ax + by$ and $\lambda y = cx + dy$ we get

$$-x = (\cos 2\theta)x + (\sin 2\theta)y$$

Looking at the previous equation

$$-x = (\cos 2\theta)x + (\sin 2\theta)y$$

and by using the identity $\cos 2\theta = 2\cos^2\theta - 1$ and the identity for $\sin 2\theta$ we arrive at

$$(-\sin\theta)y = (\cos\theta)x$$

which is the equation of the line through the origin making an angle $\theta + \pi/2$ with the x axis and is perpendicular to L.

This should not be surprising as, under reflection in L, any point P on the line L' at right angles to L is mapped onto a point P' on L'.
This is on the opposite side of O from P, where the distances OP and OP' are equal.
Because of the flip over to the other side the coordinates are multiplied by -1, the scale factor.

For a reflection of $\mathbb{R}^2$ in a line through the origin, areas are unchanged in size, but their orientation is reversed.
A triangle that has its vertices A, B, C labelled anticlockwise. Under reflection, the corresponding images A', B', C' will go round the image triangle in a clockwise direction.

A negative determinant of a matrix of a linear transformation, indicates that orientation is reversed. While a positive determinant indicates that orientation is preserved.

Stretches Parallel to the Coordinate Axes

In any of the cases where $b=c=0$, with $a=\alpha$, $d=\beta$ and $\alpha \neq \beta$ the equation

$$\lambda^2 (a+d)\lambda + (ad-bc)=0$$

becomes

$$\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta = 0$$

giving $\lambda=\alpha$ or $\lambda=\beta$, and the fixed lines are the axes of coordinates with scale factor α in the x direction and scale factor β in the y direction.

Enlargements

We next look at shears where $a=d=1$, $c=0$ and $b=k$. The above equation $\lambda^2 (a+d)\lambda + (ad-bc)=0$ then becomes

$$\lambda^2 - 2\lambda + 1 = 0$$

which means that $\lambda = 1$, and equations $\lambda x = ax + by$ and $\lambda y = cx + dy$ tell us that $y=0$. So the only fixed line through the origin is the x axis. Every point on the x-axis is a fixed point. In this case, all lines parallel to the x-axis will be fixed lines and all points on one of these fixed lines will undergo a translation parallel to the x-axis. But the magnitude of that translation will be proportional to the distance of the line from the x-axis.

Eigenvectors only tell us which are the fixed lines through the origin. Under a shear, shapes may change. But areas are presened, as expected with the matrix having a determinant of 1.

Singular Transformations

The matrices of these transformations are singular matrices. That is they have a zero determinant.

If the matrix is $\begin{pmatrix} pa & pb \\ qa & qb \end{pmatrix}$ then the equation $\lambda^2 (a+d)\lambda + (ad-bc)=0$ becomes

$$\lambda^2 - (pa+qb)\lambda = 0$$

since $paqb - qapb = 0$. If $pa+qb \neq 0$, the solutions are

$$\lambda = 0 \quad \text{and} \quad \lambda = pa+qb$$

and by substituting into $\lambda x = ax+by$ and $\lambda y = cx+dy$ we find the eigenvalue $\lambda = 0$ corresponds to an eigenvector in the direction of the line $ax+by=0$.
Eigenvalue $\lambda = pa+qb$ corresponds to the line $py=qx$. These lines are not necessarily perpendicular.
In the case $pa+qb=0$, all points are mapped onto the origin.

Linear Transformations of $\mathbb{R}^3$

As eigenvalues and eigenvectors have been defined for an $\mathbb{R}^n$ space their use in $\mathbb{R}^3$ is much the same as for $\mathbb{R}^2$.

If v is an eigenvector of t whose matrix is A, and if λ is the eigenvalue then

$$Av = \lambda v \quad \text{or written another way} \quad (A - \lambda I)v = 0$$

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Fixed lines, Eigenvectors and Eigenvalues

(continued) Linear transformation of $\mathbb{R}^3$

With linear algebra theory, this tells us that $(A-\lambda I)v=0$ only has as non-zero vector solution if the determinant of $(A-\lambda I)$ is zero.

Or that $|A-\lambda I|=0$,

This is because, if the determinant were non-zero, the matrix $(A-\lambda I)$ would have an inverse. That would mean if we were to multiply both sides of $(A-\lambda I)v=0$ on the left by this invers we should get

$$v=(A-\lambda I)^{-1}0=0$$

But v cannot be the zero vector. Since by definition an eigenvector is a non-zero vector.

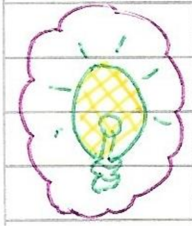
Definition

Definition 6

$|A-\lambda I|=0$ is called the characteristic equation of the matrix A.

By solving $|A-\lambda I|=0$ for λ, we can substitute each solution for λ back into the matrix equation $(A-\lambda I)v=0$.

We can then find a corresponding eigenvector, and these eigenvectors will give the directions of the fixed lines through the origin.



- In $\mathbb{R}^2$ it is easy to show that $|A-\lambda I|=0$ is equivalent to $\lambda^2-(a+d)\lambda+(ad-bc)=0 \rightarrow$

If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $|A-\lambda I| = \left|\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right|$

so that $(A-\lambda I)v=0$ becomes $(a-\lambda)(d-\lambda)-bc=0$

which, when multiplied out becomes $\lambda^2-(a+d)\lambda+ad-bc=0$

- It is still true that the product of the eigenvalues is equal to the determinant of the linear transformations. But this value now represents the scale factor

Special Cases in $\mathbb{R}^3$

Special Cases in $\mathbb{R}^3$

① Rotation about an Axis through the origin

In this case the axis is a line of fixed points, so 1 is an eigenvalue with any vector along this axis being a matching eigenvector

The other eigenvalues will be complex, so there will be no other eigen vector directions.

However since a rotation keeps volume and orientation fixed, the 3x3 matrix of the rotation will have determinant 1.

② Reflection in a Plane through the Origin

The eigenvalues for a reflection in a plane through the origin in $\mathbb{R}^3$ will be 1, 1 and −1. This is because there will be a whole plane of fixed points, and any vector along the line through the origin perpendicular to this fixed plane will be reflected.

This reflection will be to a vector of the same length but in the opposite direction to the original vector.

The eigenvalue 1 will correspond to all the vectors in the plane of reflection.

The eigenvalue −1 will correspond to vectors along the line through the origin, perpendicular to this plane of reflection.

The determinant of the matrix for this reflection is −1, and equivalenty the product of the eigenvalues is −1.

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③ Stretches, Enlargements and Shears

Special cases in $\mathbb{R}^3$ continued

Again these will be analogous to the two dimensional cases.
If the linear transformation t has eigenvalues of α, β, γ which correspond to eigenvectors of u, v, w.
Then t is equivalent to streches by scale factors α, β, γ in the directions (not necessarily orthogonal) of u, v, w.
In each case, the product of the eigenvalues will be equal to the determinant of the matrix of t.
Enlargements will involve three equal eigenvalues, and any vector in $\mathbb{R}^3$ will be an eigenvector of an enlargement centered at the origin.

- There may be shears in one or two directions, the latter transforming a rectangular block into a parallelepiped.

Example Suppose t is the linear transformation of $\mathbb{R}^3$ with matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

Show that the line $x = y = z$ is a fixed line of t, and that the scale factor of the transformation on this line is 6.
Show also that the transformation acts as an enlargement with scale factor 3 on the plane $x+y+z=0$.

Solution

Part 1 ⚬ Any point on the line $x=y=z$ has position vector of form $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$

So, $\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} k \\ k \\ k \end{pmatrix} = \begin{pmatrix} 4k+k+k \\ k+4k+k \\ k+k+4k \end{pmatrix} = 6\begin{pmatrix} k \\ k \\ k \end{pmatrix}$ So $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$ is an eigenvector of t with an eigenvalue of 6.

Part 2 ⚬ Considering the plane $x+y+z=0$, so $z = -(x+y)$ any point on it will have position vector $\begin{pmatrix} x \\ y \\ -(x+y) \end{pmatrix}$ and

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ -(x+y) \end{pmatrix} = \begin{pmatrix} 4x+y-(x+y) \\ x+4y-(x+y) \\ x+y-4(x+y) \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ -(x+y) \end{pmatrix}$$

So $x=y=z$ is a fixed line along which the scale factor of t is 6.

Every vector on the plane is an eigenvector with eigenvalue of 3. t enlarges with scale factor 3

Fixed lines, Eigenvectors and Eigenvalues — Vectors

Summary

- A function $t: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation only if it is a matrix transformation of the form $t(v) = Av$
  - where A is a real $m \times n$ matrix

- Examples of linear transformations of $\mathbb{R}^2$ (or $\mathbb{R}^3$) are rotations about the origin, reflections in lines (or planes) through the origin, stretches parallel to axes, enlargements, shears, projections or combinations of these.
  The identity transformation leaves every point fixed.

- Suppose t is a linear transformation of $\mathbb{R}^n$ with corresponding matrix A and suppose that
  $$Av = \lambda v$$
  then v is an eigenvector of t and λ is it's corresponding eigenvalue.

- The Characteristic equation of a square matrix A is $|A - \lambda I| = 0$ and the solutions to this are the eigenvalues of A.

- Eigenvectors of a linear transformation determine the directions of the fixed lines through the origin under that transformation. The corresponding eigenvalues give the scale factor of the transformation along these fixed lines.

- The product of the eigenvalues of a linear transformation t of $\mathbb{R}^n$ is equal to the determinant of the matrix of t

Graphs in the plane can be regarded as vector-valued functions even when they have been given the form $y = f(x)$. For both x and y are functions of x.

Thus any point on the curve whose equation is $y = f(x)$ has position vector $x\mathbf{i} + f(x)\mathbf{j}$.

For example the position vector of any point on the parabola whose equation is $y = x^2$ is of the form $x\mathbf{i} + x^2\mathbf{j}$. It is more practical to use the vectors $\mathbf{i}$ and $\mathbf{j}$ here, since we may have quite complicated functions involved. - Writing those as column vectors could get somewhat messy.

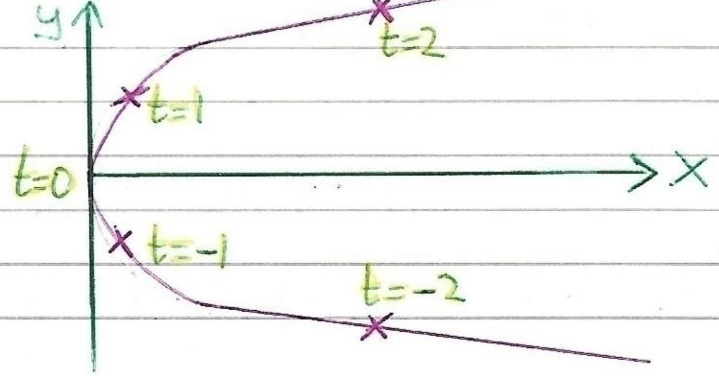
Alternatively a curve in $\mathbb{R}^2$ can be given in terms of a parameter different from x. Most will be familiar with the parametric definition of a parabola $y^2 = 4ax$ as $x = at^2$, $y = 2at$. In this case a point on the curve has position vector $at^2\mathbf{i} + 2at\mathbf{j}$.

It is sometimes convenient to think of the parameter as indicating time. As a count-down timer ticks to zero, we can think of something traveling along one half of a parabola. As the time moves on, so does the object a long the parabola.

In this way, as $t \to +\infty$ the corresponding point travels off to infinity along the upper half of the curve.

As $t \to -\infty$ the point travels off to infinity along the lower half of the curve. In this example for each point on the curve there is exactly one value of t which is its parameter.

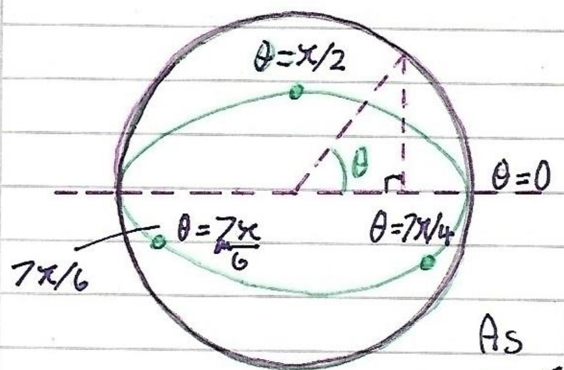
Points on the curve given by the values $t=0$, $t=\pm 1$, and $t=\pm 2$ are shown below.



Time is not the only parameter we could consider. If we think of the point given by the parametric equations:

$$x = a\cos\theta \qquad y = b\sin\theta$$

for an ellipse. Then we think of θ as being the angle shown below.



In this case although each value of θ corresponds to only one point on the curve, each point on the curve corresponds to an infinite number of θ.

As the parameter $\theta + 2n\pi$ for any $n \in \mathbb{Z}$ will give the same point on the curve as parameter θ.

Differentiation of Vectors and derived vectors in $\mathbb{R}^2$.

From now on in this section functions f, g and h will be continuous, and continuously differentiable at least as far as the second derivative.

Consider a curve C given in terms of a parameter t by the vector equation

$$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j}$$

and consider the point P on C whose parameter is t.

Suppose we increase t by a small amount δt; then the point Q with this parameter will have position vector $\mathbf{r} + \delta\mathbf{r}$, where

$$\mathbf{r} + \delta\mathbf{r} = f(t+\delta t)\mathbf{i} + g(t+\delta t)\mathbf{j}$$

Subtracting $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j}$ from the above we get:

$$\delta\mathbf{r} = \{f(t+\delta t) - f(t)\}\mathbf{i} + \{g(t+\delta t) - g(t)\}\mathbf{j}$$

repeated for clarity

$$\left(\delta\mathbf{r} = \{f(t+\delta t) - f(t)\}\mathbf{i} + \{g(t+\delta t) - g(t)\}\mathbf{j}\right)$$

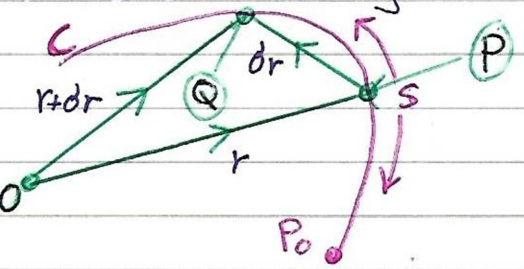
(Continued) Dividing both sides of the bottom of last page's equation, we get:

$$\frac{\delta r}{\delta t} = \frac{f(t+\delta t)-f(t)}{\delta t} i + \frac{g(t+\delta t)-g(t)}{\delta t} j$$

and as we allow δt to tend to zero, the right side above tends to

$$\frac{df}{dt} i + \frac{dg}{dt} j$$

But looking at the diagram below, as we allow δt to tend to zero, the vector δr approaches the direction of the tangent and $\delta r/\delta t$ will approach dr/dt - the rate of change of position vector which is in essence a velocity vector.



- **Definition 1**

- If a position vector is given in terms of a parameter α, so that

$$r = f(\alpha) i + g(\alpha) j$$

then the vector $dr/d\alpha$ given by

$$\frac{dr}{d\alpha} = \frac{df}{d\alpha} i + \frac{dg}{d\alpha} j$$

is called the derived vector or tangent vector of r with respect to α.

There are two parameters for which the derivatives have standard abbreviations. The first is the case where the parameter t represents time, and we write:

$$\dot{f} = \frac{df}{dt} \quad \text{and} \quad \dot{g} = \frac{dg}{dt}$$

(continued) If

$$\dot{f} = \frac{df}{dt} \text{ and } \dot{g} = \frac{dg}{dt}$$

was the first case of using standard abbreviations. Then the second case is differentiation with respect to x, where we write:

$$f'(x) = \frac{df}{dx} \text{ and } g'(x) = \frac{dg}{dx}$$

We used the term tangent vector previously. As in ordinary calculus, derivatives help us to find tangents.

Suppose we consider the parabola again, and let us take a particular value of a.

Say $a = \frac{1}{2}$, then a general point on this parabola has position vector

$$r = \tfrac{1}{2}t^2 i + t j$$

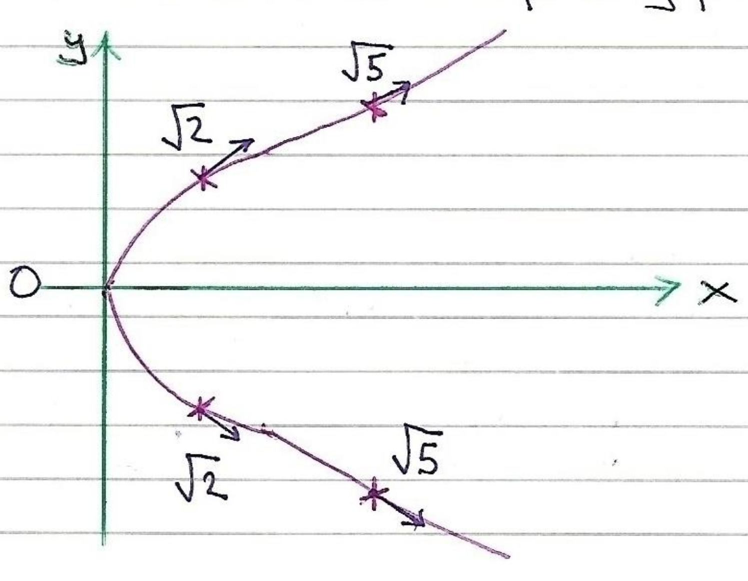
Now if we differentiate this vector with respect to t component-wise we get the derived vector:

$$\dot{r} = t i + j$$

If we draw the direction of the above vector at the point whose parameter is t on the diagram below - we see that the derived vector is parallel to the tangent to the curve at that point.

By considering the particular values of t we picked, we confirm this tangency property by drawing the derived vectors at these points.

The magnitudes (given by $\sqrt{1+t^2}$) of the derived vectors are written near the corresponding points.



If the position vector $\mathbf{r}$ of the point P is given in terms of a time parameter t. Then $\dot{\mathbf{r}}$ gives the velocity vector at this particular time.

Being a vector this means it has both magnitude and direction. So in the previous page's example we see that as t increases from $-\infty$ to $+\infty$, the particle describing the curve continuously slows down until it reaches the $t=0$ point.

Then it speeds up again ever increasing as $t \to \infty$.

If we differentiate again, we see that:

$$\ddot{\mathbf{r}} = \mathbf{i} + 0\mathbf{j} = \mathbf{i}$$

and the acceleration seen to be constant, with no component on the y-direction, and 1 unit in the positive x-direction.

Curves in three dimensions

In three dimensions a curve can be defined parametrically by a single variable as follows:

$$\mathbf{r} = f(\alpha)\mathbf{i} + g(\alpha)\mathbf{j} + h(\alpha)\mathbf{k}$$

and the derived vector will be $\dfrac{d\mathbf{r}}{d\alpha} = \dfrac{df}{d\alpha}\mathbf{i} + \dfrac{dg}{d\alpha}\mathbf{j} + \dfrac{dh}{d\alpha}\mathbf{k}$

Example 1

Consider the curve given by $\mathbf{r} = a(\cos)t\,\mathbf{i} + a(\sin)t\,\mathbf{j} + bt\mathbf{k}$ where a and b are constants.

This is the equation of a helix.

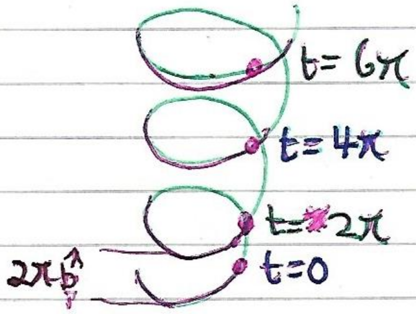
If $b=0$, the curve is simply a circle in the plane $z=0$, and as t increases from $-\infty$ to ∞ so the point with parameter t continues to describe the circle infinitely many times.

However, if $b>0$ as t increases so does the height since

$$\dot{\mathbf{r}} = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + b\mathbf{k}$$

(continued) Since $\ddot{r} = -a \sin t\, i + a \cos t\, j + bk$ the point with parameter t moves upwards at constant speed on a helix described on a vertical circular cylinder of radius a - shown below.

If $b < 0$ then we would still obtain a helix, but one that would be a reflection in the plane $z=0$, of the helix below.



Rules For Differentiating Vectors

In differentiation of scalar functions we have the addition rule, the product rule, the quotient rule, and so on.

With vectors we have an addition rule, and since there are three sorts of product involving vectors - the is a rule for each. However they all follow the same basic rules.

Rule 1: If c is a constant vector $\frac{dc}{dt} = 0$

Rule 2: $\frac{d}{dt}(u+v) = \frac{du}{dt} + \frac{dv}{dt}$

Rule 3: If ϕ is a scalar function $\frac{d}{dt}(\phi u) = \frac{d\phi}{dt} u + \phi \frac{du}{dt}$

Rule 4: $\frac{d}{dt}(u.v) = u.\frac{dv}{dt} + \frac{du}{dt}.v$

Rule 5: $\frac{d}{dt}(u \times v) = u \times \frac{dv}{dt} + \frac{du}{dt} \times v$

Example 2

- We shall show that rule 4 on the previous page:

$$\frac{d}{dt}(u.v) = u.\frac{dv}{dt} + \frac{du}{dt}.v$$

holds for:

$$u = f_1(t)i + g_1(t)j + h_1(t)k \text{ and } v = f_2(t)i + g_2(t)j + h_2(t)k$$

So:
$$\frac{du}{dt} = \frac{df_1}{dt}i + \frac{dg_1}{dt}j + \frac{dh_1}{dt}k$$

$$\frac{dv}{dt} = \frac{df_2}{dt}i + \frac{dg_2}{dt}j + \frac{dh_2}{dt}k$$

and that:

$$u.\frac{dv}{dt} + \frac{du}{dt}.v = f_1\frac{df_2}{dt} + g_1\frac{dg_2}{dt} + h_1\frac{dh_2}{dt} + \frac{df_1}{dt}f_2 + \frac{dg_1}{dt}g_2 + \frac{dh_1}{dt}h_2$$

$$\rightarrow = f_1\frac{df_2}{dt} + \frac{df_1}{dt}f_2 + g_1\frac{dg_2}{dt} + \frac{dg_1}{dt}g_2 + h_1\frac{dh_2}{dt} + \frac{dh_1}{dt}h_2$$

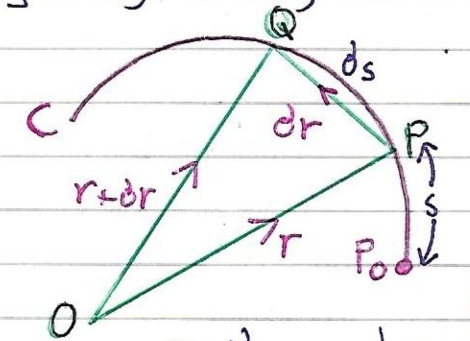
$$\rightarrow = \frac{d}{dt}(f_1f_2 + g_1g_2 + h_1h_2)$$

$$\rightarrow = \frac{d}{dt}(u.v)$$

The Serret-Frenet equations for a curve in $\mathbb{R}^3$

One of the useful parameters for a point P on a curve is the distance s of the point P from a fixed point P_0 measured along the curve.

Looking at the diagram, we can see that not only does δr approach the tangent in direction as $\delta t \to 0$. But also the length $|\delta r|$ approaches δs, - the length of arc from P to Q.



This means that in the limiting case dr/ds is a Unit vector.

- Suppose then we call this unit vector t; then t is the unit tangent vector to the curve at P.

Since t is a unit vector $t.t = 1$, which is a constant, and so by that rule(1) and rule 4 which says

$$\frac{d}{dt}(U.V) = U.\frac{dV}{dt} + \frac{dU}{dt}.V$$

we have

$$(\text{zero})\, 0 = \frac{d}{ds}(t.t) = t.\frac{dt}{ds} + \frac{dt}{ds}.t = 2t.\frac{dt}{ds}$$

From this equation we find that

$$t.\frac{dt}{ds} = 0$$

and provided $dt/ds \neq 0$ means that dt/ds is perpendicular to t.

Let n denote the unit vector in the direction of dt/ds so that:

$$\frac{dt}{ds} = \kappa n$$

for some scalar function κ of s.

'κ' is called the curvature of C at the point with parameter s.

'n' is called the unit principal normal to the curve.

(continued) - In fact, if γ is a circle which touches C at P, which has the same curvature as C at P, and lies in the plane through P parallel to both ~~t~~ t and n. Then the radius of γ will be $\rho = 1/k$, and we call ρ the radius of curvature of C at P.

Now t and n are orthogonal unit vectors. So their vector product is a unit vector also, we call this:

$$b = t \times n$$

the unit binormal of C at P.

Because t, n, and b are all unit vectors which are mutually orthogonal, By the arguments described previous that led to

$$t \cdot \frac{dt}{ds} = 0$$

we have
$$t \cdot \frac{dt}{ds} = n \cdot \frac{dn}{ds} = b \cdot \frac{db}{ds} = 0$$
and because of mutual orthogonality

becomes
$$t \cdot n = n \cdot b = b \cdot t = 0.$$

If we now apply rule 5 from earliear:

$$\frac{d}{dt}(u \times v) = u \times \frac{dv}{dt} + \frac{du}{dt} \times v.$$

we get

$$\frac{db}{ds} = t \times \frac{dn}{ds} + \frac{dt}{ds} \times n = t \times \frac{dn}{ds}$$

Since n is parallel to dt/ds, and hence their vector product is zero. Taking the vector product of n with each side of this equation -

$$n \times \frac{db}{ds} = n \times \left(t \times \frac{dn}{ds}\right) = \left(n \cdot \frac{dn}{ds}\right) t - (n \cdot t) \frac{dn}{ds} = 0$$

- By $\left[t \cdot \frac{dt}{ds} = n \cdot \frac{dn}{ds} = b \cdot \frac{db}{ds} = 0\right]$ and $\left[t \cdot n = n \cdot b = b \cdot t = 0\right]$

Meaning for some scalar function τ of s $\left(\frac{db}{ds} = -\tau n\right)$ holds true provided $db/ds \neq 0$ and n and db/ds are not parallel.

we call this the torsion of C at P.

(continued) Since t, n, and b are mutually orthogonal unit vectors, and since $b = t \times n$. We can deduce that $n = b \times t$.

By differentiating both sides of this we obtain

$$\frac{dn}{ds} = b \times \frac{dt}{ds} + \frac{db}{ds} \times t = \kappa b \times n - \tau n \times t$$

that is

$$\frac{dn}{ds} = \tau b - \kappa t$$

By collecting together this last equation with two we previously had -

$$\frac{dt}{ds} = \kappa n \quad \text{and} \quad \frac{db}{ds} = -\tau n$$

we are able to find:

$$\frac{dt}{ds} = \kappa n, \quad \frac{dn}{ds} = \tau b - \kappa t, \quad \frac{db}{ds} = -\tau n$$

These are the Serret-Frenet equations.

If a curvature κ is large at a point P, then the curve is very bent near that point. If κ is small, then the curve approaches straightness near point P.

If $\kappa = 0$ at a point P, then we say the curve is straight at that point.

The above doesn't actually mean the curve is a straight line close to P (although that is one possibility) but P may be a point where the curve bends one way on one side of P, and another on the other side.

We can compare this with a point of inflection in calculus, where the second derivative of the function zero, since If $\kappa = 0$, from the Serret-Frenet equations:

$0 = \frac{dt}{ds} = \frac{d^2r}{ds^2}$ - If $\kappa = 0$ for the whole curve, then the whole curve is a straight line.

(continued) What happens if $\tau=0$?
In this case **b** is stationary at the point, so there is no 'twist' in the curve at the point.
If $\tau=0$ for the whole curve, then the curve lies entirely within a plane. So the curvature κ measures how 'bent' the curve is, and the torsion τ measures how 'twisted' the curve is.

Going back to the helix example that was defined by the equation:

$$\mathbf{r} = (a\cos\theta)\mathbf{i} + (a\sin\theta)\mathbf{j} + b\theta\mathbf{k}$$

we find that

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds} = \left\{(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}\right\}\frac{d\theta}{ds}$$

Now we know that $d\mathbf{r}/ds$ is a unit vector, so that

$$\sqrt{a^2\sin^2\theta + a^2\cos^2\theta + b^2}\ \frac{d\theta}{ds} = 1$$

and therefore $\dfrac{d\theta}{ds} = \dfrac{1}{\sqrt{a^2+b^2}} = c$, for example.

Now $\mathbf{t} = d\mathbf{r}/ds$, so that: $\mathbf{t} = c\left\{(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}\right\}$

which means: $\kappa\mathbf{n} = \dfrac{d\mathbf{t}}{ds} = -ca\left\{(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}\right\}\dfrac{d\theta}{ds}$

and since **n** is a unit vector, $\kappa = c^2 a$ which gives

$$\kappa = \frac{a}{a^2+b^2} \quad \text{and} \quad \mathbf{n} = (-\cos\theta)\mathbf{i} + (-\sin\theta)\mathbf{j}$$

we know that $\mathbf{b} = \mathbf{t}\times\mathbf{n}$, so by differentiating this we get

$$\frac{d\mathbf{b}}{ds} = \frac{b}{a^2+b^2}\left\{(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}\right\}$$

$$= -\frac{b}{a^2+b^2}\mathbf{n}$$

And from the 3rd Serret-Frenet equation we then get $\boxed{\tau = \dfrac{b}{a^2+b^2}}$ So we have found that

$$\tau = \frac{b}{a^2+b^2}$$

for the helix previous, both torsion and curvature are constant at all points

Differentiation of vectors and derived vectors

Vectors

Summary

① Curves in $\mathbb{R}^3$ can be defined parametrically by a single variable as follows:

$$r = f(\alpha)i + g(\alpha)j + h(\alpha)k$$

(in $\mathbb{R}^2$ there is simply $h(\alpha)=0$)

② The derived vector of r above with respect to α will be

$$\frac{dr}{d\alpha} = \frac{df}{d\alpha}i + \frac{dg}{d\alpha}j + \frac{dh}{d\alpha}k$$

③ Rules for differentiating sums and products follow the usual pattern for derivatives of sums and products of vector-valued functions, and these can be found in section 1 of this part of the text.

④ The Secret-Frenet equations involve the unit tangent, normal and bi-normal vectors to the curve, **t**, **n** and **b** respectively. These are:

$$\frac{dt}{ds} = \kappa n, \quad \frac{dn}{ds} = \tau b - \kappa t, \quad \frac{db}{ds} = -\tau n$$

⑤ In the above equations, the arc length from a given point on the curve is measured by 's'.

κ (tiny capital 'K') is called the curvature

τ is called the torsion

and $\rho = 1/\kappa$ is the 'radius of curvature at the point concerned,

Vector Differentiation

With differentiation of real valued functions of one variable we have:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

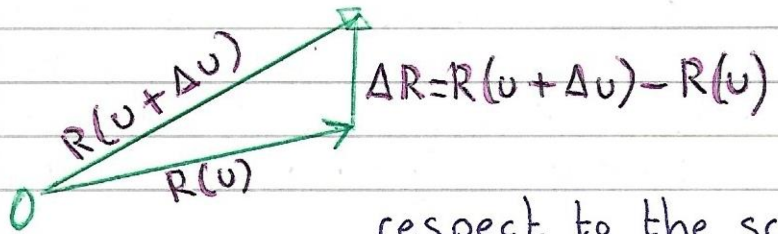
We now extend this to vector valued functions of a single variable.

Ordinary Derivatives of Vector-Valued Functions

Suppose $R(u)$ is a vector depending on a single scalar variable u. Then

$$\frac{\Delta R}{\Delta u} = \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

where Δu denotes an increment in u as shown below



The ordinary derivative of the vector $R(u)$ with respect to the scalar u is given as follows when the limit exists:

$$\frac{dR}{du} = \lim_{\Delta u \to 0} \frac{\Delta R}{\Delta u} = \lim_{\Delta u \to 0} \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

- Since dR/du is itself a vector depending on u, we can consider its derivative with respect to u. If this derivative exists, we denote it by d^2R/du^2. Similary, higher-order derivatives are described.

Motion: Velocity and Acceleration

Suppose particle P moves along a space curve C whose parametric equations are $x = x(t)$, $y = y(t)$, $z = z(t)$ where t = time. Then position vector of particle P along the curve is $r(t) = x(t)i + y(t)j + z(t)k$.

(continued) In such a case as the previous page, Velocity and acceleration of the particle P is given by

$$v = v(t) = \frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

$$a = a(t) = \frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$$

Space Curves

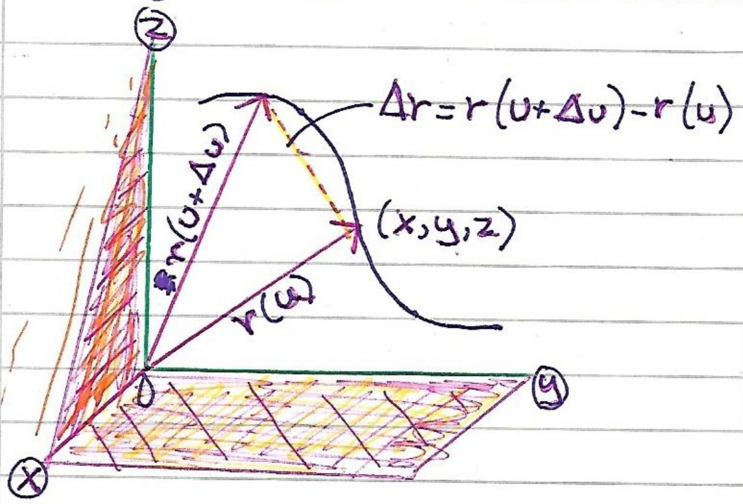
Consider the position vector $r(u)$ joining the origin O of a coordinate system and any point (x,y,z). So

$$r(u) = x(u)i + y(u)j + z(u)k$$

and the specification of the vector function $r(u)$ defines x, y and z as functions of u. As u changes, the terminal point of r describes a space curve having parametric equations

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

Then the following is a vector in the direction of Δr if $\Delta u > 0$ and in the direction of $-\Delta r$ if $\Delta u < 0$.



$$\frac{\Delta r}{\Delta u} = \frac{r(u+\Delta u) - r(u)}{\Delta u}$$

Suppose

$$\lim_{\Delta u \to 0} \frac{\Delta r}{\Delta u} = \frac{dr}{du}$$

exists...

Then the limit will be a vector in the direction of the tangent to the space curve at (x,y,z) and it is given by:

$$\frac{dr}{du} = \frac{dx}{du}i + \frac{dy}{du}j + \frac{dz}{du}k$$